On the Theory of Abelian Groups

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Abstract

In this expository paper,[1] we discuss the completeness and decidability of the theory of abelian groups, the theory of divisible torsion free abelian groups, and the theory of groups with elements of order at most 2.

1 Introduction

In this section, we introduce the different collections of abelian groups that we will discuss throughout the paper. Using some basic linear algebra, we will be able to characterize some of these collections. Even more, we'll show that some of the theories corresponding to these collections of groups are decidable. Together with Godel's Completeness Theorem, this tells us that we have an effective procedure to figure out if any statement (written in a relatively basic language) about such a group is "true."

1.1 Abelian Groups

Let us recall that $(G, +)$ is an *abelian group* if $+ : G \times G \to G$ is an operation satisfying the following conditions:

- *Associativity*: we have $x + (y + z) = (x + y) + z$ for any $x, y, z \in G$.
- *Identity*: there exists an identity $0 \in G$ so that $x + 0 = x$ for any $x \in G$.
- *Inverse*: for every $x \in G$, there exists $y \in G$ such that $x + y = 0$.
- *Commutativity*: $x + y = y + x$ for every $x, y \in G$.

Since our standard definition of a group has a very "axiomatic" nature, it's quite easy to give a formal set of axioms for the theory of abelian groups. Let $\mathcal{L}$ be a language with equality, one binary function symbol $+$, and one constant symbol 0. For notational convenience, we will write $x + y$ instead of $+xy$. Also, let us write $x \neq y$ for $\neg x = y$. Allow Σ_{ab} to consist of the following sentences:

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

$$\forall x (x + 0 = x)$$

[1] We assume some elementary knowledge of logic, linear algebra, and group theory.

$$\forall x \exists y (x + y = 0)$$

$$\forall x \forall y (x + y = y + x)$$

Then, it's clear that $(G, +, 0)$ is a model for Σ_{ab} iff it's an abelian group.[2] Throughout the paper, we will use the notation $T_{\text{ab}} := \text{Cn}\,\Sigma_{\text{ab}}$ and $\mathcal{K}_{\text{ab}} := \text{Mod}\,T_{\text{ab}}$, so that T_{ab} is the theory of abelian groups and $\mathcal{K}_{\text{ab}}$ is the collection of all abelian groups.

1.2 Divisible Torsion Free Abelian Groups

Besides discussing the theory of abelian groups, will also explore some specific subcollections of abelian groups satisfying some additional properties. Remember that we say $x \in G$ has *order* $|x| = n \in \mathbb{Z}^+$ if $nx = 0$ and $kx \neq 0$ for $1 \leq k < n$ where

$$mx := \underbrace{x + x + \ldots + x}_{m \text{ times}}$$

We say that $x \in G$ is *torsion free* if $nx \neq 0$ for all $n \in \mathbb{Z}^+$ and that G is *torsion free* if x is torsion free for all $x \in G \setminus \{0\}$. Also, we call a group *divisible* if for every $x \in G$ and $n \in \mathbb{Z}^+$ there exists some $y \in G$ such that $ny = x$. To familiarize ourselves with these definitions, let's see some examples.

Example 1. $(\mathbb{Q}, +)$ *and* $(\mathbb{R}, +)$ *are divisible torsion free abelian groups.*

Example 2. *The Klein Group* $V = \langle a, b \mid a + a = b + b = (a + b) + (a + b) = 0 \rangle$ *is not torsion-free. In fact, every element but the identity has order* 2*.*

Example 3. $(\mathbb{C}^\times, \cdot)$ *is not torsion free since* $i^4 = 1$*. That said, it contains torsion free elements such as* 2 *(note that* $2^n \neq 1$ *for all* $n \in \mathbb{Z}^+$*). It's a divisible group since it's closed under taking* n*-th roots.*

Example 4. $(\mathbb{Z}, +)$ *is a torsion free abelian group that fails to be divisible.*

As we did with abelian groups, let's axiomatize the theory of divisible torsion free abelian groups. Let

$$\Sigma_{\text{tf}} = \{\forall x (x \neq 0 \rightarrow nx \neq 0)\}_{n=1}^{\infty}$$

$$\Sigma_{\text{d}} = \{\forall x \exists y (x = ny)\}_{n=1}^{\infty}$$

Note that a group is torsion free iff it's a model for Σ_{tf} and is divisible iff it's a model for Σ_{d}. Hence, G is a divisible torsion free abelian group iff G is a model of $\Sigma_{\text{ab}} \cup \Sigma_{\text{tf}} \cup \Sigma_{\text{d}}$. Throughout

[2] Note that here we are using the symbols $+$ and 0 for both the syntax and the semantics. Fortunately, this notational ambiguity won't led to any issues.

the paper, we will write $T_{\mathrm{tf}} := \mathrm{Cn}\,(\Sigma_{\mathrm{ab}} \cup \Sigma_{\mathrm{tf}})$ and $\mathcal{K}_{\mathrm{tf}} = \mathrm{Mod}\,T_{\mathrm{tf}}$ so that T_{tf} is the theory of torsion free abelian groups and $\mathcal{K}_{\mathrm{ab}}$ is the collection of all torsion free abelian groups. Also, we will write $T_{\mathrm{dtf}} := \mathrm{Cn}\,(\Sigma_{\mathrm{ab}} \cup \Sigma_{\mathrm{tf}} \cup \Sigma_{\mathrm{d}})$ and $\mathcal{K}_{\mathrm{dtf}} = \mathrm{Mod}\,T_{\mathrm{dtf}}$ so that T_{dtf} is the theory of divisible torsion free abelian groups and $\mathcal{K}_{\mathrm{ab}}$ is the collection of all divisible torsion free abelian groups.

1.3 Groups satisfying $x + x = 0$

We'll also discuss the collection of groups satisfying that every element in the group has order at most 2. That is, the collection of models of $\Sigma_{\mathrm{ab}} \cup \Sigma_2$ where $\Sigma_2 = \{\forall x(x + x = 0)\}$. Throughout the paper, we will write $T_2 := \mathrm{Cn}\,\Sigma_2$ and $\mathcal{K}_2 = \mathrm{Mod}\,T_2$ so that T_2 is the theory of abelian groups with elements of order at most 2 and $\mathcal{K}_2$ is the collection of such groups.

Example 5. *For any $n \in \mathbb{Z}^+$, we have*

$$\underbrace{(\mathbb{Z}/2\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/2\mathbb{Z})}_{n \text{ times}} \in \mathcal{K}_2$$

2 An Underlying Vector Space Structure

2.1 Groups satisfying $x + x = 0$ as $\mathbb{F}_2$-vector spaces

Let $G \in \mathcal{K}_2$. Then, we can think of G as a vector space over $\mathbb{F}_2 = \{\bar{0}, \bar{1}\}$ where the scalar multiplication $\cdot$ is given in the obvious way: $\bar{0} \cdot x = 0$ for all $x \in G$ and $\bar{1} \cdot x = x$. To show this, we need to verify the vector space axioms:

- *Identity as a scalar*: $\bar{1} \cdot x = x$ for all $x \in G$.
- *Compatibility*: $(ab) \cdot x = a \cdot (b \cdot x)$ for all $x \in G$ and $a, b \in \mathbb{F}_2$.
- *Distributivity over scalars*: $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $x \in G$ and $a, b \in \mathbb{F}_2$.
- *Distributivity over vectors*: $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $x, y \in G$ and $a \in \mathbb{F}_2$.

Thanks to the simplicity of our definition of scalar multiplication, it is not hard to check that $(G, +, \cdot)$ is a vector space over $\mathbb{F}_2$. The only non-trivial check is that $(\bar{1} + \bar{1}) \cdot x = \bar{1} \cdot x + \bar{1} \cdot x$. However, this follows from the fact that $x + x$ for any $x \in G$ since

$$(\bar{1} + \bar{1}) \cdot x = \bar{0} \cdot x = 0 = x + x = \bar{1} \cdot x + \bar{1} \cdot x$$

It turns out that any $G \in \mathcal{K}_{\mathrm{dtf}}$ also has a natural vector space structure. However, this won't be as easy to define nor to verify. In fact, in order to show this, we will first prove that any $G \in \mathcal{K}_{tf}$ has an induced $\mathbb{Z}$-module structure.

2.2 Torsion free abelian groups as $\mathbb{Z}$-modules

Fix $G \in \mathcal{K}_{\text{tf}}$. We have already defined mx for $m \in \mathbb{Z}_{\geq 0}$ and $x \in G$. Note that we can extend this definition to any $m \in \mathbb{Z}$ by letting $mx = (-m)(-x)$ where $-x$ is the unique[3] inverse of x for any $m < 0$. We will show that this operation, which will sometimes denote by $*$ and sometimes by no symbol at all, gives us a $\mathbb{Z}$-module structure on G. Recall that a module is like a vector space, but over a ring instead of a field (see [1], page 337, for a formal definition). We need to prove that $* : \mathbb{Z} \times G \to G$ satisfies

- *Identity as a scalar*: $1 * x = x$ for all $x \in G$.
- *Compatibility*: $(ab) * x = a * (b * x)$ for all $x \in G$ and $a, b \in \mathbb{Z}$.
- *Distributivity over scalars*: $(a + b) * x = a * x + b * x$ for all $x \in G$ and $a, b \in \mathbb{Z}$.
- *Distributivity over "vectors"*: $a * (x + y) = a * x + a * y$ for all $x, y \in G$ and $a \in \mathbb{Z}$.

First, we prove the following lemma:

Lemma 2.1. *For any $x \in G$ and $n \in \mathbb{Z}$, we have $-(nx) = n(-x) = (-n)x$.*

Proof: Note that the second equality is immediate from our definition of $*$ so it suffices to prove the first equality. If $n \geq 0$, all we are saying is that $nx + n(-x) = 0$, which is clear. If $n < 0$, then

$$-(nx) = -((-n)(-x)) = (-n)x = n(-x)$$

by the lemma for positive integers. □

Proposition 2.2. *$(G, +, *)$ is a module over $\mathbb{Z}$.*

Proof: $1x = x$ follows immediately from the definition of $*$. If $a, b \in \mathbb{Z}_{\geq 0}$, the compatibility of $*$ is straightforward. This suffices, since we know that we can push the $-$ symbols to the right by Lemma 2.1. As an example, we show the case in which $a < 0 \leq b$:

$$\begin{aligned}(ab)x &= (-(-a)b)x \\ &= ((-a)b)(-x) \\ &= (-a)(b(-x)) \\ &= a(bx)\end{aligned}$$

Since both distributivities are clear from the definition of $*$, we are done. □

[3] This is an elementary result of group theory. To see this for abelian groups, note that if y and z are both inverses for x, then $y = (x + z) + y = (x + y) + z = z$.

2.3 Divisible torsion free abelian groups as $\mathbb{Q}$-vector spaces

Fix $G \in \mathcal{K}_{\text{dtf}}$. Let us show that we can think of G as a vector space over $\mathbb{Q}$. It turns out that to even defining the scalar multiplication will be challenging.

Lemma 2.3. *For each $x \in G$ and $n \in \mathbb{Z} \setminus \{0\}$, there exists a unique $y \in G$ such that $x = ny$.*

Proof: Suppose we have $y, z \in G$ such that $ny = x = nz$. Then, $ny + n(-z) = 0$ where $-z$ is the unique inverse element of z. Since G is abelian, we get that $n(y + (-z)) = 0$. This tells us that $y + (-z) = 0$ since we assumed G is torsion free so we conclude that $y = z$. □

Lemma 2.3 allows us to make the following definition:

Definition 2.1. *We say that $y \in G$ is the divisor of $x \in G$ by $n \in \mathbb{Z} \setminus \{0\}$ if y is the unique element such that $x = ny$. In this case, we write $y := \frac{x}{n}$.*

Now, let $\phi : \mathbb{Z} \times \mathbb{Z}^+ \times G \to G$ be given by

$$\phi(p, q, x) = p\,\frac{x}{q}$$

Let us show that ϕ induces a map $\mathbb{Q} \times G \to G$. To see this, we need to check that

$$\frac{p_1}{q_2} = \frac{p_2}{q_2} \implies \phi(p_1, q_1, x) = \phi(p_2, q_2, x)$$

for all $x \in G$, $p_i \in \mathbb{Z}$, and $q_i \in \mathbb{Z}^+$. Assume the antecedent, so that $p_1 q_2 = p_2 q_1$. Then,

$$\begin{aligned} q_1 q_2\, \phi(p_1, q_1, x) &= q_1 q_2 p_1 \frac{x}{q_1} \\ &= q_2 p_1\, x \\ &= p_2 q_1\, x \\ &= q_1 q_2\, p_2 \frac{x}{q_2} \\ &= q_1 q_2\, \phi(p_2, q_2, x) \end{aligned}$$

by an implicit use of the compatibility of $*$. Since Lemma 2.3 gives us that $ax = ay \implies x = y$, we get that $\phi(p_1, q_1, x) = \phi(p_2, q_2, x)$. Hence, this induces a map $\mathbb{Q} \times G \to G$ given by

$$\frac{p}{q} \cdot x = \phi(p, q, x)$$

Let us show that this map acts as scalar multiplication. To do so, we'll need the following easy lemma.

Lemma 2.4. *We have*

$$\frac{x}{nm} = \frac{\left(\frac{x}{n}\right)}{m}$$

for any $x \in G$ and $n, m \in \mathbb{Z}^+$.

Proof: By Lemma 2.3, it suffices to see that

$$nm\,\frac{x}{nm} = x = n\,\frac{x}{n} = nm\,\frac{\left(\frac{x}{n}\right)}{m}$$

□

Proposition 2.5. $(G, +, \cdot)$ *is a vector space over* $\mathbb{Q}$.

Proof: We need to show that the vector space axioms are satisfied by $(G, +, \cdot)$. Let us first prove that

$$\left(\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}\right)\cdot x = \frac{p_1}{q_1}\cdot\left(\frac{p_2}{q_2}\cdot x\right)$$

Thus, we need to prove that

$$p_1p_2\,\frac{x}{q_1q_2} = p_1\left(\frac{p_2\,\frac{x}{q_2}}{q_1}\right) \tag{1}$$

By Lemma 2.3, it suffices to show that Equation 1 mutiplied by q_1q_2 on both sides holds. Note that

$$\begin{aligned} q_1q_2p_1p_2\,\frac{x}{q_1q_2} &= p_1p_2\,x \\ &= q_2p_1p_2\,\frac{x}{q_2} \\ &= q_1q_2p_1\left(\frac{p_2\,\frac{x}{q_2}}{q_1}\right) \end{aligned}$$

Clearly,

$$\frac{1}{1}\cdot x = \frac{x}{1} = x$$

so all we are left to do is to show distributivity. Note that

$$q\,\frac{x+y}{q} = x + y = q\,\frac{x}{q} + q\,\frac{x}{q} = q\left(\frac{x}{q} + \frac{x}{q}\right)$$

Hence, we get that

$$\frac{x+y}{q} = \frac{x}{q} + \frac{x}{q}$$

by Lemma 2.3. Then, Proposition 2.2 gives us that

$$\frac{p}{q}\,(x+y) = p\,\frac{x+y}{q}$$

$$\begin{aligned}
&= p\left(\frac{x}{q}+\frac{y}{q}\right)\\
&= p\,\frac{x}{q}+p\,\frac{y}{q}\\
&= \frac{p}{q}\,x+\frac{p}{q}\,y
\end{aligned}$$

Lastly, let us prove distributivity over scalars. Note that

$$\begin{aligned}
\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right)\cdot x &= \frac{p_1q_2+p_2q_1}{q_1q_2}\cdot x\\
&= (p_1q_2+p_2q_1)\cdot\frac{x}{q_1q_2}\\
&= p_1q_2\,\frac{x}{q_1q_2}+p_2q_1\,\frac{x}{q_1q_2}\\
&= p_1q_2\,\frac{\left(\frac{x}{q_1}\right)}{q_2}+p_2q_1\,\frac{\left(\frac{x}{q_2}\right)}{q_1}\\
&= p_1\,\frac{x}{q_1}+p_2\,\frac{x}{q_2}\\
&= \frac{p_1}{q_1}\cdot x+\frac{p_2}{q_2}\cdot x
\end{aligned}$$

With this, we conclude that $(G,+,\cdot)$ is a vector space over $\mathbb{Q}$. □

3 Categoricity

Lemma 3.1. *T_{dtf} is κ-categorical for any $\kappa>\aleph_0$.*

Proof: Let $G,H\in\mathcal{K}_{\text{dtf}}$ be such that $|G|=\mathfrak{c}=|H|$. By Proposition 2.5, we know that G and H are vector spaces over $\mathbb{Q}$. Let β be a basis for G and γ be a basis for H. Note that $|\beta|=\kappa=|\gamma|$ since $\mathbb{Q}$ is countable and hence $|\text{span}(S)|=|S|$ for any infinite set S. Thus, there exists a bijection $\psi:\beta\to\gamma$. We know from linear algebra that we can extend any bijection of bases to an isomorphism of vector spaces. Thus, G and H are isomorphic as vector spaces and therefore also as groups. □

Since the only finite torsion free abelian groups is the trivial group,[4] we know that T_{dtf} is n-categorical (vacuously) for any $n\in\mathbb{Z}_{\geq 0}$. However, it turns out that T_{dtf} is not κ-categorical for any cardinal κ, as shown by the following claim:

Claim 3.2. *T_{dtf} is not $\aleph_0$-categorical.*

[4] This will be formally shown later on, see Lemma 4.7.

Proof: Note that $\mathbb{Q}, \mathbb{Q}^2 \in \mathcal{K}_{\text{dtf}}$. Thus, they are both models for T_{dtf}. Let us prove that $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic as abelian groups and hence that T_{dtf} is not $\aleph_0$-categorical. Suppose by contradiction that $\mathbb{Q} \simeq \mathbb{Q}^2$ and that φ was such an isomorphism. Since $\mathbb{Q}$ has dimension 1 as a vector space over $\mathbb{Q}$, we have a non-trivial pair $(a,b) \in \mathbb{Q}^2$ such that $a\,\varphi(e_1) + b\,\varphi(e_2) = 0$ for a basis $\{e_1, e_2\}$ of $\mathbb{Q}^2$. Then,

$$\begin{aligned} 0 &= \varphi^{-1}(0) \\ &= \varphi^{-1}(a\,\varphi(e_1) + b\,\varphi(e_2)) \\ &= a\,\varphi^{-1}(\varphi(e_1)) + b\,\varphi^{-1}(\varphi(e_2)) \\ &= ae_1 + be_2 \end{aligned}$$

This is absurd since we assumed that $\{e_1, e_2\}$ was a basis for $\mathbb{Q}^2$ and that $(a,b) \neq 0$. We conclude that $\mathbb{Q} \not\simeq \mathbb{Q}^2$. □

Note that the the divisibility condition is essential in the proof of Lemma 3.1:

Claim 3.3. *T_{tf} is not $\mathfrak{c}$-categorical for $\mathfrak{c} = |\mathbb{R}|$.*

Proof: Consider $G = \{f : \mathbb{Z} \to \mathbb{Z}\}$ with pointwise addition. Then, $|G| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Note that $G \not\simeq \mathbb{R}$ since $\mathbb{R}$ is divisible while G is not. Hence, T_{tf} is not $\mathfrak{c}$-categorical. □

T_{dtf} is $\mathfrak{c}$-categorical despite that T_{tf} is not because divisible torsion free abelian groups have a vector space structure while torsion free abelian groups only have a module structure. This difference is important because for $\mathbb{Z}$-modules (that are not finitely generated) we cannot say much about their bases (if they even exist). Fortunately, the same argument we used to show Lemma 3.1 gives us a similar result for T_2:

Lemma 3.4. *T_2 is κ-categorical for any infinite cardinal κ.*

Proof: Let G and H be groups with elements of order at most 2 such that $|G| = \kappa = |H|$. We've shown that G and H are vector spaces over $\mathbb{F}_2$. Let β and γ be a basis for G and H, respectively. Since span(S) is finite for any finite S, we must have $|\beta| = \kappa = |\gamma|$. Hence, there's a bijection $\beta \to \gamma$ that can be extended to a vector space isomorphism. In particular, G and H are isomorphic as groups. □

4 Completeness and Decidability

Two of the most interesting properties we can ask our theories to satisfy are those of completeness and decidability.

Remark 4.1. *T_{ab} is not complete since for $\tau = \exists x(x + x = 0)$ we have $(\mathbb{F}_2, +) \models \tau$ but $(\mathbb{R}, +) \not\models \tau$.*

Remark 4.2. *T_2 is not complete since if we let τ be the sentence that asserts the existence of only 2 elements, then $\mathbb{Z}/2\mathbb{Z} \models \tau$ but $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \not\models \tau$.*

Now, let

$$\Sigma_\infty = \{\exists v_1 \exists v_2 \ldots \exists v_n (v_1 \neq v_2 \wedge \ldots \wedge v_1 \neq v_n \wedge v_2 \neq v_3 \wedge \ldots \wedge v_2 \neq v_n \wedge \ldots \wedge v_{n-1} \neq v_n)\}_{n=1}^{\infty}$$

and define $T_{2\infty} = \mathrm{Cn}(\Sigma_{\mathrm{ab}} \cup \Sigma_2 \cup \Sigma_\infty)$ and let $\mathcal{K}_{2\infty} = \mathrm{Mod}\, T_{2\infty}$. Although Remark 4.2 shows that T_2 is incomplete, if we also ask its models to be infinite, then the resulting theory is indeed complete, as shown by the following theorem:

Theorem 4.3. *$T_{2\infty}$ is complete.*

In order to prove the theorem, let us first recall the celebrated Łoś–Vaught Test.

Theorem 4.4 (Łoś–Vaught Test). *Let T be a theory in a countable language with no finite models. Then, if T is κ-categorical for some infinite cardinal κ, then T is complete.*

The proof of the Łoś–Vaught Test can be found in Enderton's textbook (page 157 of [2]). Now, the proof of Theorem 4.3 is reduced to putting two pieces together:

Proof (Theorem 4.3): This follows immediately from the Łoś–Vaught Test and Lemma 3.4. □

Corollary 4.5. *$T_{2\infty}$ is decidable.*

Proof: Recall that any complete, axiomatized theory is decidable. However, note that this theory is axiomatized by a decidable collection of sentences: $\Sigma_{\mathrm{ab}} \cup \Sigma_2 \cup \Sigma_\infty$. □

Remark 4.6. *T_{dtf} is not complete since for $\tau = \exists x(x \neq 0)$ we have $(\mathbb{R}, +) \models \tau$ but $(\{0\}, +) \not\models \tau$.*

In some way, Remark 4.6 is misleading since simply adding the condition $\exists x(x \neq 0)$ to T_{dtf} suffices to make it complete. To prove it, we first need the following lemma:

Lemma 4.7. *If $0 \neq G \in \mathcal{K}_{\mathrm{dtf}}$, then $|G| = \infty$.*

Proof: Let $0 \neq G \in \mathcal{K}_{\mathrm{dtf}}$. Since $G \neq 0$, there exists $x \in G$ such that $x \neq 0$. Note that $nx \neq mx$ for all $m < n$: otherwise, we would have $(n-m)x = 0$, contradicting the assumption of G being torsion free. Hence, $\{nx\}_{n=1}^{\infty}$ is an infinite subset of G and hence G is infinite. □

Let $T_{\mathrm{dtf}\infty} = \mathrm{Cn}(\Sigma_{\mathrm{ab}} \cup \Sigma_{\mathrm{d}} \cup \Sigma_{\mathrm{tf}} \cup \{\exists x(x \neq 0)\})$ and let $\mathcal{K}_{\mathrm{dtf}\infty} = \mathrm{Mod}\, T_{\mathrm{dtf}\infty}$.

Theorem 4.8. *$T_{\mathrm{dtf}\infty}$ is complete.*

Proof: This follows immediatealy from the Łoś–Vaught Test, Lemma 3.1, and Lemma 4.7. □

Corollary 4.9. *$T_{\mathrm{dtf}\infty}$ is decidable.*

Proof: Note that $\Sigma_{\mathrm{ab}} \cup \Sigma_{\mathrm{tf}} \cup \Sigma_{\mathrm{d}} \cup \{\exists x(x \neq 0)\}$ is decidable. □

References

[1] David Steven Dummit and Richard M Foote. *Abstract Algebra*, volume 3. Wiley Hoboken, 2004.

[2] Herbert B Enderton. *A Mathematical Introduction to Logic*. Elsevier, 2001.