

A Prime-Representing Constant

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Abstract. We present a constant and a recursive relation to define a sequence f_n such that the floor of f_n is the n th prime. Therefore, this constant generates the complete sequence of primes. We also show this constant is irrational and consider other sequences that can be generated using the same method.

In this note we present a constant and a recursive relation that generates the complete sequence of primes.

Theorem 1. *Let p_n denote the n th prime. Then there exists a constant*

$$f_1 = 2.920050977316\dots$$

and a sequence

$$f_n = \lfloor f_{n-1} \rfloor (f_{n-1} - \lfloor f_{n-1} \rfloor + 1)$$

such that the floor of f_n is the n th prime, i.e., $\lfloor f_n \rfloor = p_n$.

Proof. Define a sequence g_n as follows:

$$g_n = \sum_{k=1}^n \frac{p_k - 1}{\prod_{i=1}^{k-1} p_i}$$

Bertrand's postulate gives bounds on the size of each prime, namely $p_{n-1} < p_n < 2p_{n-1} - 1$. We use this as we consider g_n as n tends to infinity:

$$\begin{aligned} \lim_{n \rightarrow \infty} g_n &= (p_1 - 1) + \frac{p_2 - 1}{p_1} + \frac{p_3 - 1}{p_2 p_1} + \frac{p_4 - 1}{p_3 p_2 p_1} + \dots \\ &< (p_1 - 1) + \frac{2p_1 - 2}{p_1} + \frac{2p_2 - 2}{p_2 p_1} + \frac{2p_3 - 2}{p_3 p_2 p_1} + \dots \end{aligned}$$

Most terms on the right-hand side of the inequality cancel out, leaving $(p_1 - 1) + 2 = 3$. Since g_n is strictly increasing and bounded, it is convergent. Its value is 2.920050977316...

Define $f_1 = \lim_{n \rightarrow \infty} g_n$ and

$$\begin{aligned} f_n &= (f_1 - g_{n-1}) \prod_{i=1}^{n-1} p_i \\ &= (p_n - 1) + \frac{p_{n+1} - 1}{p_n} + \frac{p_{n+2} - 1}{p_{n+1} p_n} + \frac{p_{n+3} - 1}{p_{n+2} p_{n+1} p_n} + \dots \end{aligned}$$

It can be determined that $f_n = p_{n-1}(f_{n-1} - p_{n-1} + 1)$.

Using Bertrand's postulate again, we have

$$f_n > (p_n - 1) + \frac{p_n - 1}{p_n} + \frac{p_{n+1} - 1}{p_{n+1}p_n} + \frac{p_{n+2} - 1}{p_{n+2}p_{n+1}p_n} + \dots$$

and

$$f_n < (p_n - 1) + \frac{2p_n - 2}{p_n} + \frac{2p_{n+1} - 2}{p_{n+1}p_n} + \frac{2p_{n+2} - 2}{p_{n+2}p_{n+1}p_n} + \dots$$

Most terms in the inequalities cancel out, leaving $p_n < f_n < p_n + 1$, and so the floor function $\lfloor f_n \rfloor = p_n$. The sequence f_n can now be generated recursively with the formula $f_n = \lfloor f_{n-1} \rfloor (f_{n-1} - \lfloor f_{n-1} \rfloor + 1)$. ■

There are many other examples of prime-producing formulae. One of the earliest and most famous examples of a prime-producing formula is the polynomial discovered by Euler

$$n^2 + n + 41.$$

This simple polynomial gives prime values for each integer n from 0 to 39.

There are also many examples of prime-generating constants. For example, if $\alpha = \sum_{i \geq 1} p_i / 10^{2^{i+1}}$, then α is a prime-generating constant because we may generate the n th prime with the formula

$$p_n \lfloor 10^{2^{n+1}} \alpha \rfloor - 10^{2^n} \lfloor 10^{2^n} \alpha \rfloor.$$

It is clear that α is irrational since it is given as a nonrepeating decimal. See [2] Theorem 1.2.2.

We took inspiration for Theorem 1 from Mills's constant as described by William H. Mills in 1947 in [3]. In this one-page note, Mills defines a constant A and a function $\lfloor A^{3^n} \rfloor$ in such a way that the value of the function is prime for all natural numbers n .

Mills's constant is defined as the smallest positive real number that may be used for A . Mills did not give any specific value for A ; however, if the Riemann hypothesis is true, then we may define $A = 1.306377883863\dots$, which generates the following sequence:

$$2, 11, 1361, 2521008887, 16022236204009818131831320183, \dots$$

Each number in this sequence is prime. However, this is far from a complete sequence of primes. Indeed, the next value in this sequence is of the order 10^{84} . It is not yet known whether Mills's constant is irrational.

Theorem 2. *The prime-generating constant f_1 is irrational.*

Proof. Since $p_n < f_n < p_n + 1$ for all n , we may write $f_n = p_n + r_n$, where $0 < r_n < 1$.

Assume f_1 is rational, so that $f_1 = a/b$. Using the recurrence relation

$$f_{n+1} = p_n(f_n - p_n + 1) = p_n(r_n + 1),$$

we see that bf_n is an integer for all n . In particular, $r_n \geq 1/b$ for all n .

However, we can rearrange the above expression as follows:

$$(r_n + 1) = \frac{f_{n+1}}{p_n} = \frac{p_{n+1} + r_{n+1}}{p_n}.$$

By the prime number theorem we know that p_{n+1}/p_n tends to 1 as n tends to infinity. This means that, since the r_n are bounded, the right-hand side also tends to 1. And so $\lim_{n \rightarrow \infty} r_n = 0$. This contradicts $r_n \geq 1/b$ for all n . ■

The prime-generating constant of Theorem 1 and Theorem 2 has appeared before, albeit in a different context, in [1], [4], and [5]. In these papers the constant is described as the average of the sequence 2, 3, 2, 3, 2, 5, 2, 3, . . . , the sequence of smallest primes that do not divide n .

To prove that the average of this sequence is the same as our constant, let us consider the probability that a prime p_k is the smallest prime that does not divide n , for some natural number n . This probability can be written as

$$P(p_k \text{ does not divide } n \text{ and } p_1, p_2, \dots, p_{k-1} \text{ divide } n) = \left(1 - \frac{1}{p_k}\right) \prod_{i=1}^{k-1} \frac{1}{p_i}.$$

The average of the sequence is therefore given by

$$\sum_{k=1}^{\infty} P(\text{The smallest prime that does not divide } n \text{ is } p_k) \cdot p_k = \sum_{k=1}^{\infty} \frac{p_k - 1}{\prod_{i=1}^{k-1} p_i},$$

which is the definition of our prime-generating constant in Theorem 1.

The arguments in this note are not limited to the sequence of primes. Indeed, they can be applied to any sequence that follows Bertrand’s postulate, and any such sequence will define its own constant f_1 . This constant can then be used in the same formula for f_n described in Theorem 1.

Similarly, if a sequence follows Bertrand’s Postulate and the ratio of consecutive terms tend to 1, then the sequence-generating constant is irrational by the same argument used in the proof of Theorem 2.

If we just consider the lower bound of Bertrand’s postulate, the most compact sequence we get is 2, 3, 4, 5, 6, 7, In this case the sequence defines the Euler constant, that is to say $f_1 = e$. Since the ratio of consecutive integers tends to 1 this gives us another proof that e is irrational.

At the other extreme, if we just consider the upper bound of Bertrand’s postulate, the most compact sequence we get is 3, 4, 6, 10, 18, 34, Here the n th term is $2^{n-1} + 2$ and the sequence defines the constant 3.56797609098 Interestingly, we have not found this constant mentioned anywhere in the literature before.

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